Integer Linear Programming: Introduction to Branch & Bound

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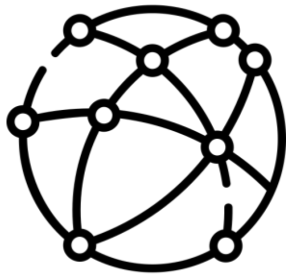
ILP definition

$$\min_x \{c^T x : Ax \leq \mathrm{b}\}, x \in \mathbb{N}^n, A \in \mathbb{R}^{m*n}, b \in \mathbb{R}^m$$

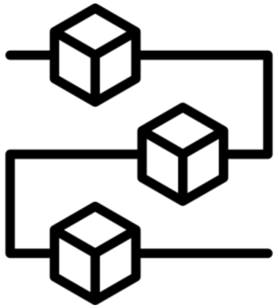
- $x = (x_i)_{1 \leq i \leq n}$: Set of **integer** variables
- $c^T x = \sum_i x_i c_i$: Minimization function
- $Ax \leq b: 1 \leq j \leq m, \sum_i A_{j,i} x_i \leq b_j$: Set of constraints

What are the applications of ILP?

Most of the problem are not LP but ILP problems



- Production planning
- Scheduling
- Networks
- …



Companies using solvers

Join Industry Leaders Who Rely on Gurobi for Smarter, Faster, Decisions

Existing solvers

<u>Commercial (need to pay ☹)</u>

- FICO Xpress (1986)
- CPLEX (1996)
- Gurobi (2008)

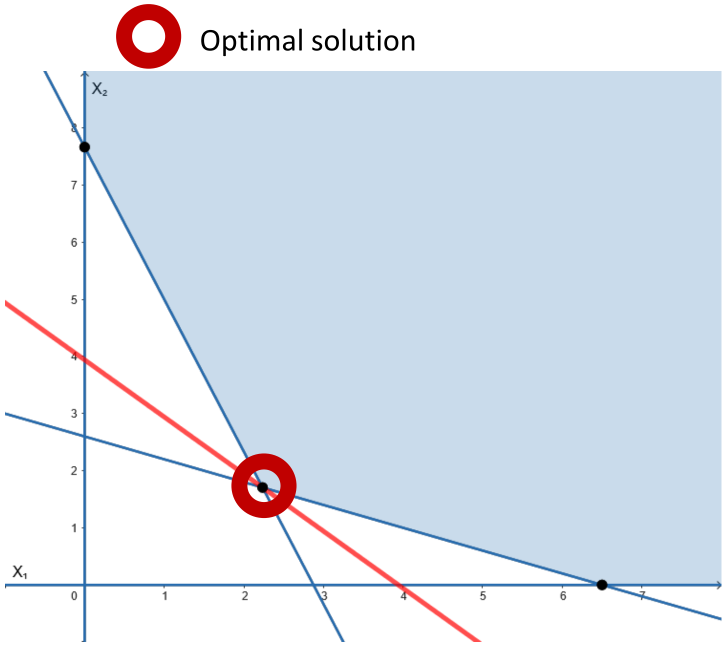
<u>Open source</u>

- SCIP (2002)
- HiGHs (2018)

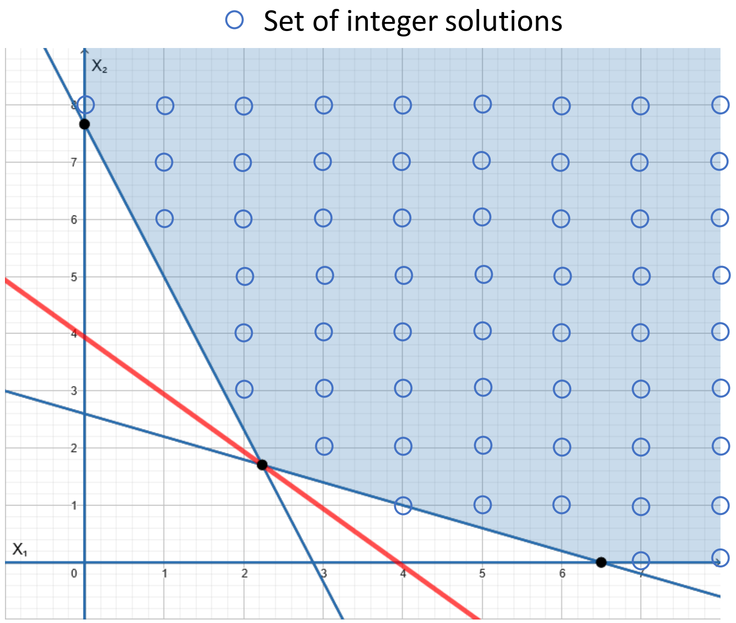
Most of them are portable on Python

LP and ILP

- Solving LP - simplex algorithm: polynomial complexity
- Solving ILP: NP-complete, exponential according to the dimensions of the problem



LP



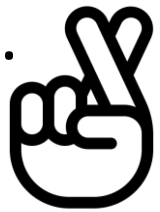
ILP

How to solve an ILP?

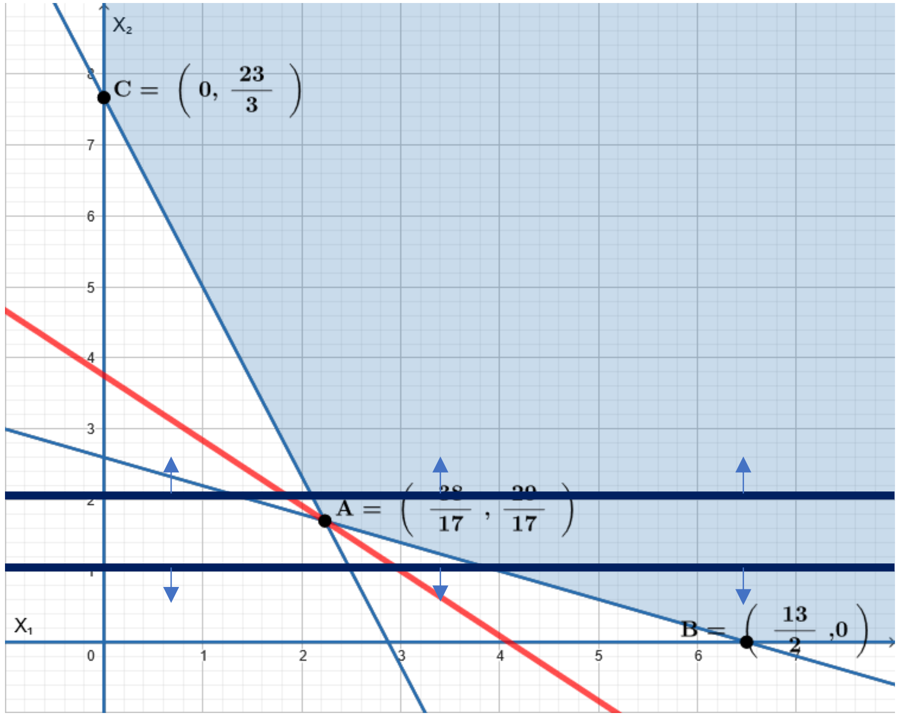
BIG IDEA:

- It exist methods to solve LP problems in polynomial time
- BUT not method to solve an ILP in polynomial time.

→We consider the "relaxed version" of the ILP, and solve the LP problem, cross the fingers that the solution gives an integer sol.

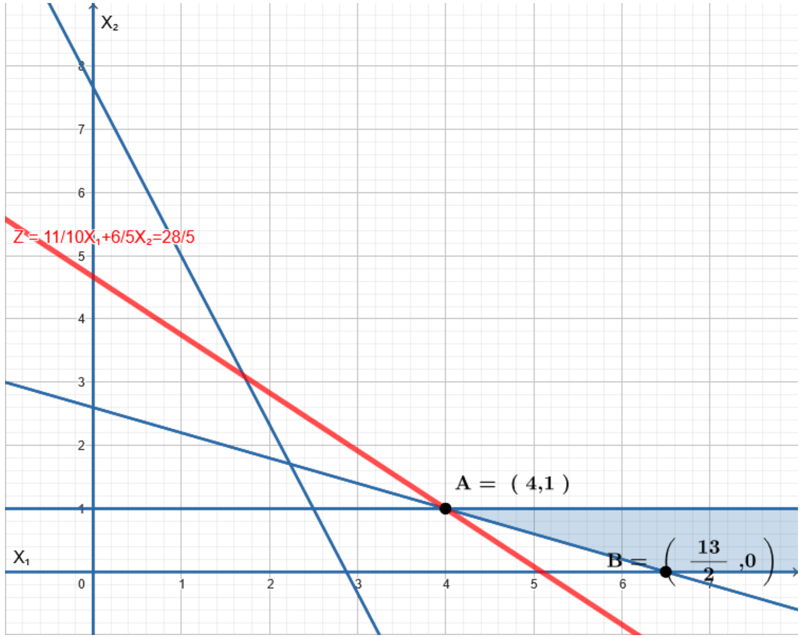


- If not, then we divide the problem in subproblems that does not include that value, and cross fingers again, etc..

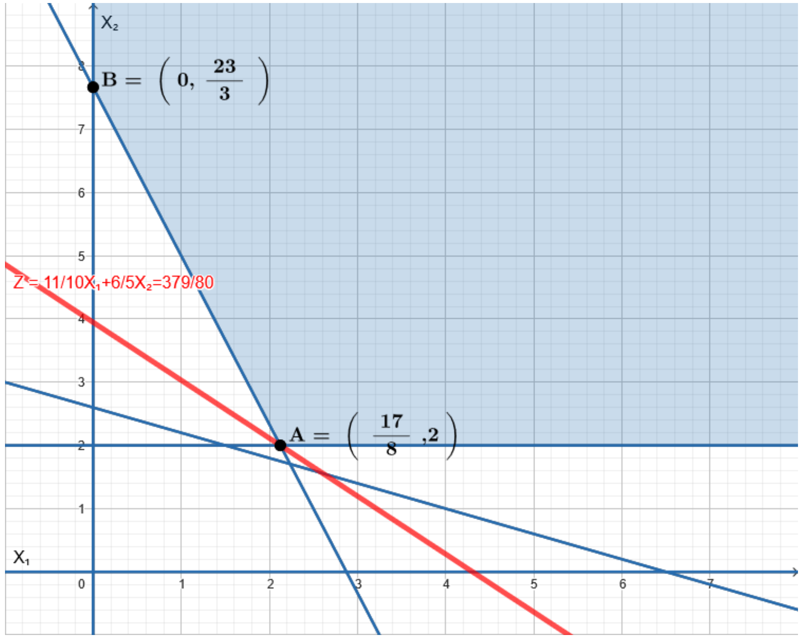


P_0

$$P_0 = P_{00} \cup P_{01}$$



P_{00}



P_{01}

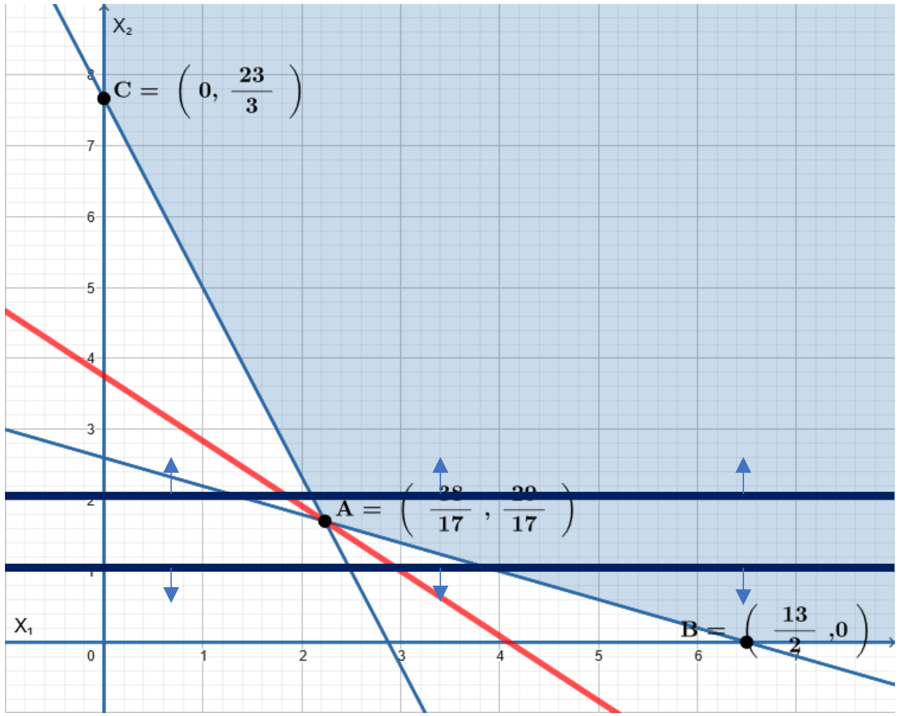
THIS IS THE BRANCH AND BOUND

Branch and Bound: *"Divide and conquer"*

Consider a problem P^0 with objective of $min_{x \in K} f(x)$, K^0 representing the constraints on the variables, f the minimization function; we consider an existing solution (the best for now) x^* leading the score $f(x^*)$.

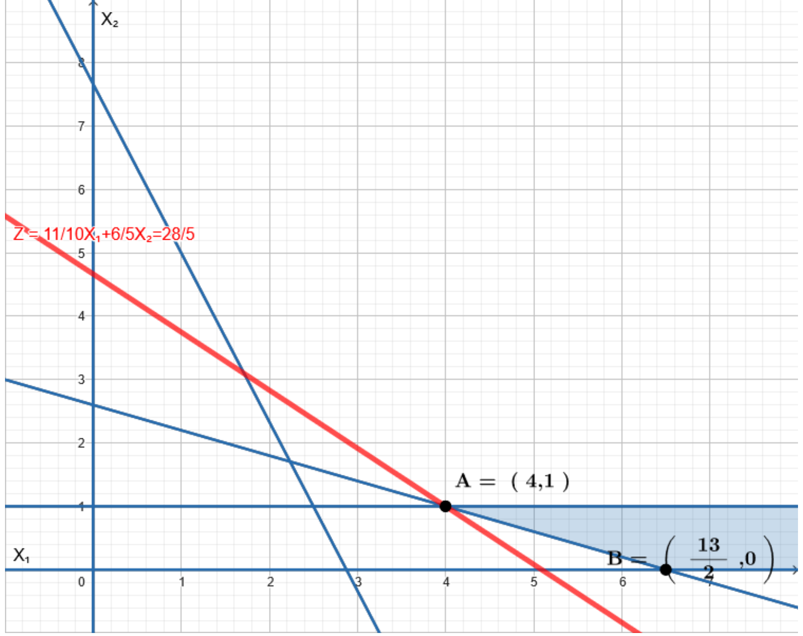
Take one subproblem P^k, solve its relaxed version to have $x^k = \left(x_i^k\right)_{i \in [1,n]}$ with score $f(x^k)$.

- if the solution is integer, x^k is the solution of P^k, so it is one solution for P. We update the best solution $f(x^*)$ if the solution is improved $f(x^*) = \min(f(x^k), f(x^*))$.
- Else:
  - *Conquer*:
    - if $f(x^k) > f(x^*)$, P^k wont lead to a better solution than $f(x^*)$, so we stop the exploration on this problem, and prune.
    - Else, we do *divide* and split the variable set P^k.
  - *Divide*: we select one of the variable x^ksuch that $x_i^k$is not integer; the variable set K^k is split into $K_{i-}^k = K^k, x_i \leq \lfloor x_i^k \rfloor$ and $\mathrm{K}_{\mathrm{i+}}^{\mathrm{k}} = K^k, x_i \geq \lceil x_i^k \rceil$ to define the problems P_{i-}^k andP_{i+}^k.



X₂
C = (0, 23/3)
A = (38/17, 29/17)
B = (13/2, 0)
X₁

P_0



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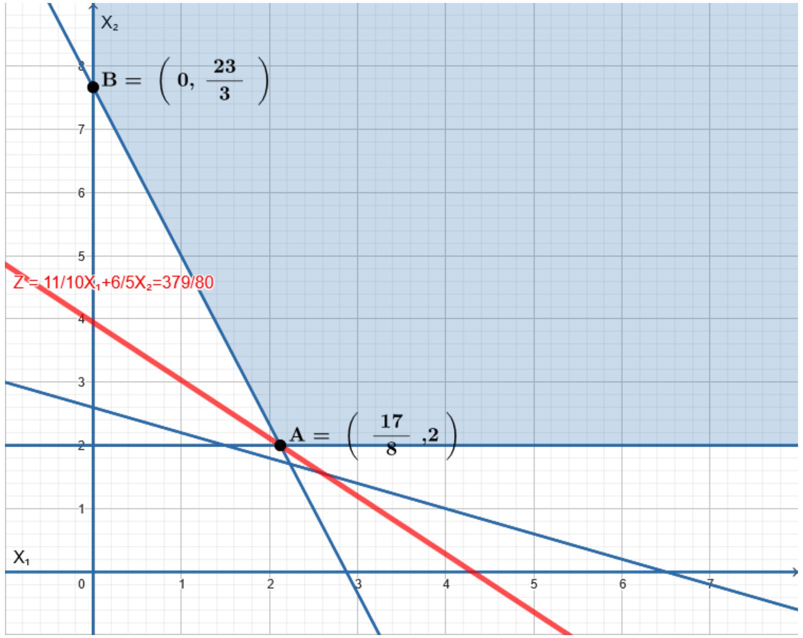
P_{00}

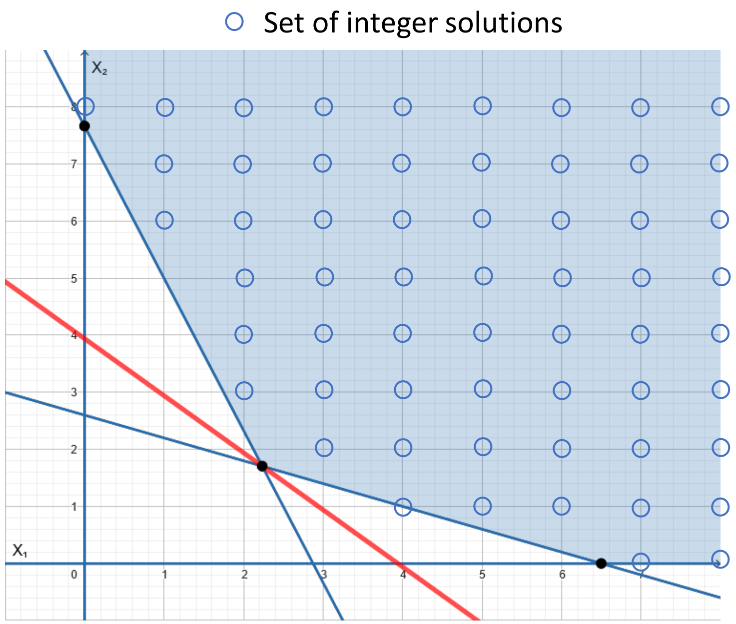


X₂
B = (0, 23/3)
Z*=11/10X₁+6/5X₂=379/80
A = (17/8, 2)
X₁

P_{01}

New trick: not exploring the unnecessary suproblems

The trick: considering a subproblem P^k, the LP relaxation give you x^k. Then for sure all the integer solution from P^k are going to be worst than $f(x^k)$.



In the case you found at least one solution, the best is x^*, and the score is $f(x^*)$.

Then, if you explore one subproblem P^k and the LP relaxation give you x^k, but $f(x^k)$ with worst than $f(x^*)$, no need to explore it, **we prune**.

Branch and Bound: *"Divide and conquer"*

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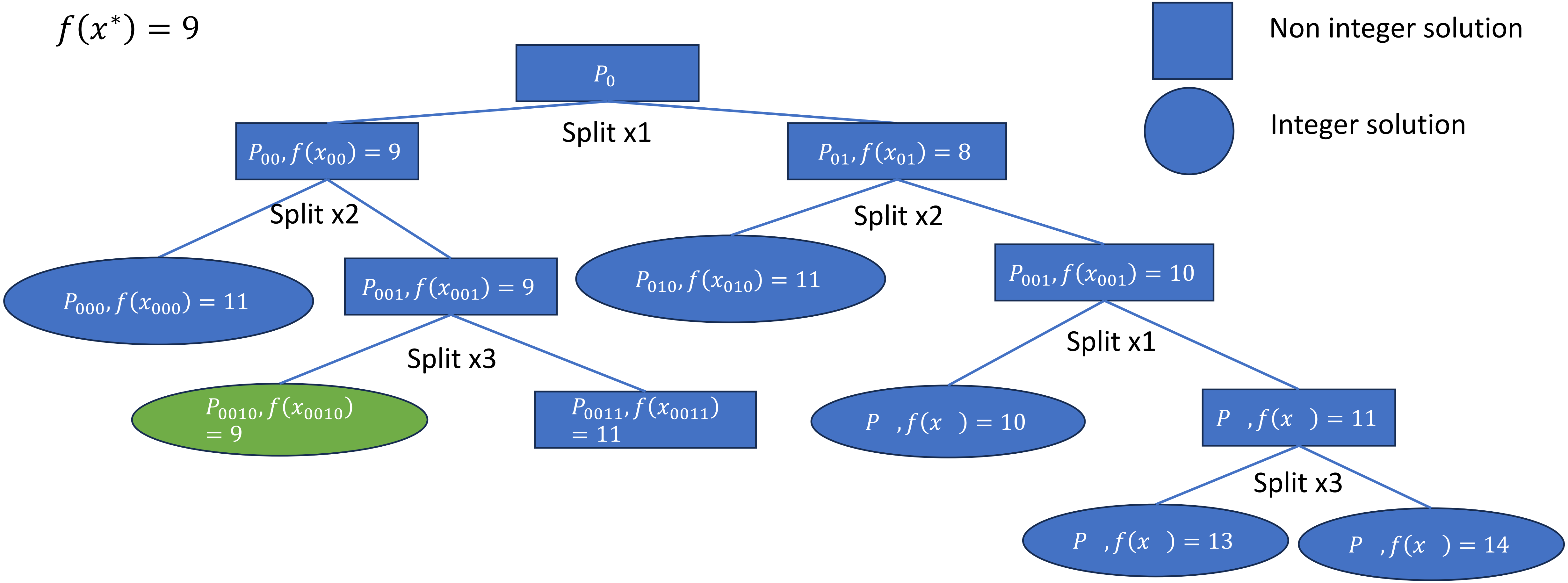
Branch and Bound: tree representation



Features of the B&B: *Branching strategy*

- *Branching strategy*: which non integer variable to select to split a problem into subproblems.

- *Search strategy*: which subproblem to explore the first.

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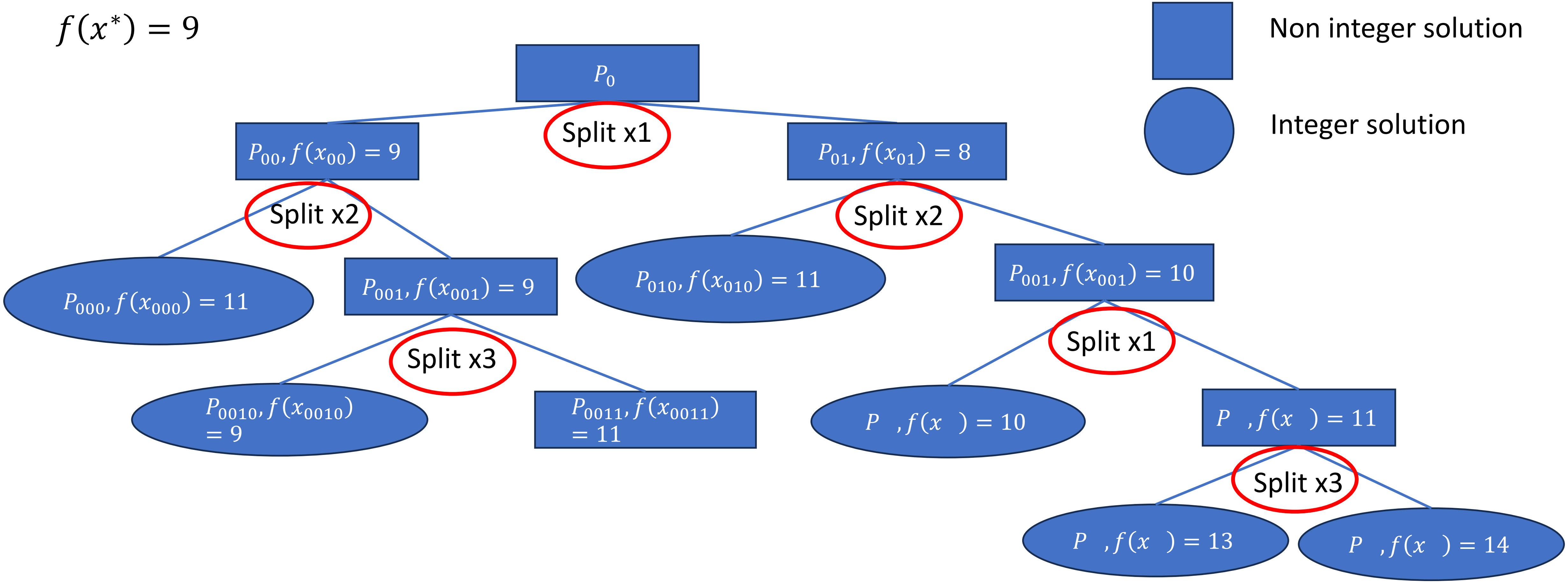
Branching strategy : tree representation

$f(x^*) = 9$



Features of the B&B: *Branching strategy*

- *Branching strategy*: which non integer variable to select to split a problem into subproblems.

→ <u>one method</u>: according to the solution x of the relaxed problem, select the variable the closest to 0.5 in the decimals.

- *Search strategy*: which subproblem to explore the first.

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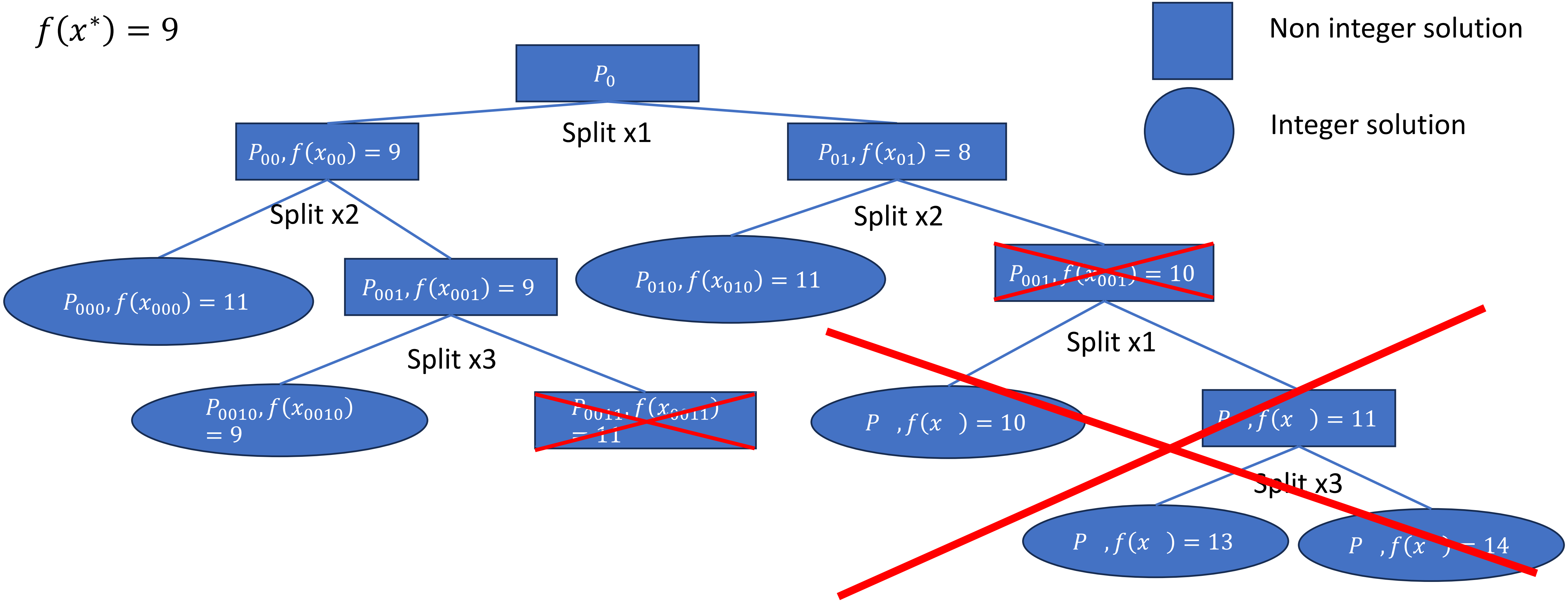
Features of the B&B: *Search strategy*

- *Branching strategy*: which non integer variable to select to split a problem into subproblems

- *Search strategy*: which subproblem to explore the first

→one method: always select the problem with the lowest lower-bound

Search strategy : tree representation



Example

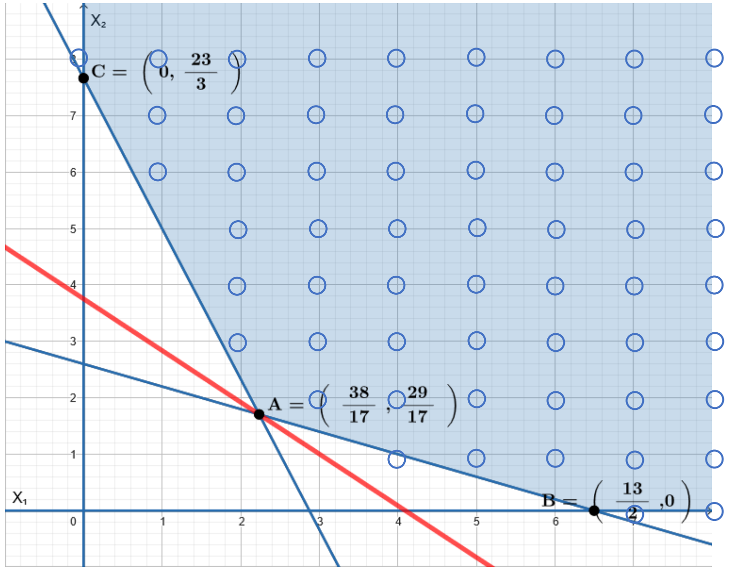
minimize $z = 1.1x_1 + 1.2x_2$

subject to $2x_1 + 5x_2 \geq 13$

$8x_1 + 3x_2 \geq 23$

$x_1, x_2 \geq 0$

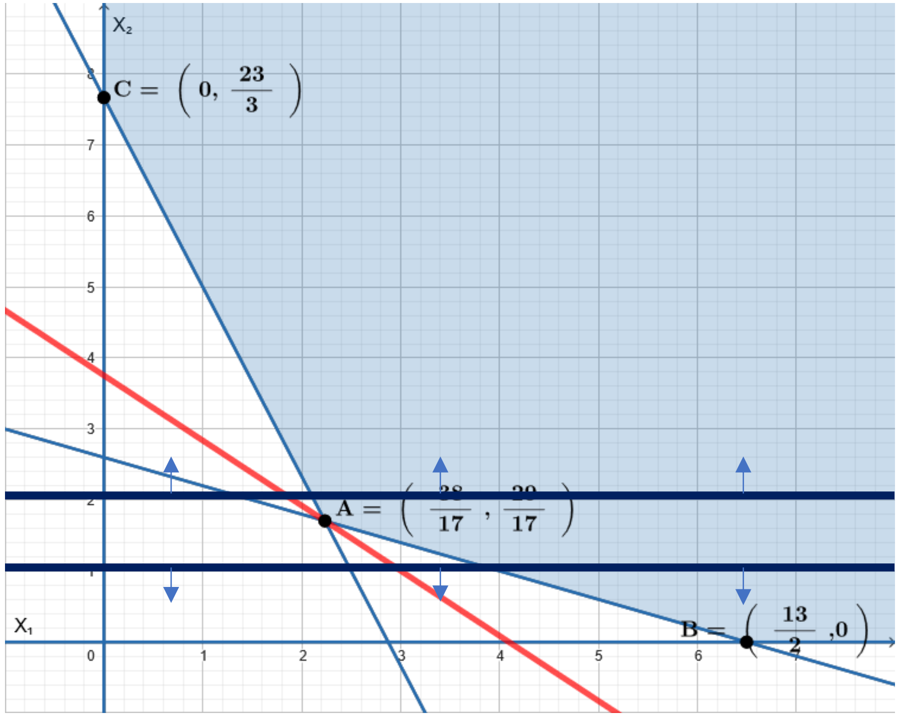
$x_1, x_2 \in \mathbb{N}$



○ Set of integer solutions

Divide step 1

- The best solution of P_0 is X1=2.24 , X2=1.71.
- Divide P_0 according to variable x_2. Solve graphically the subproblems P_{00} and P_{01}.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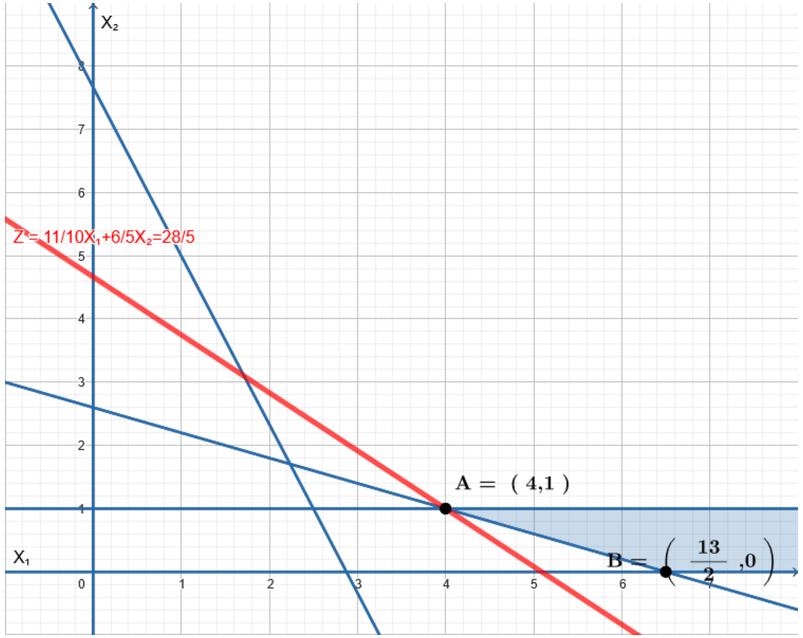
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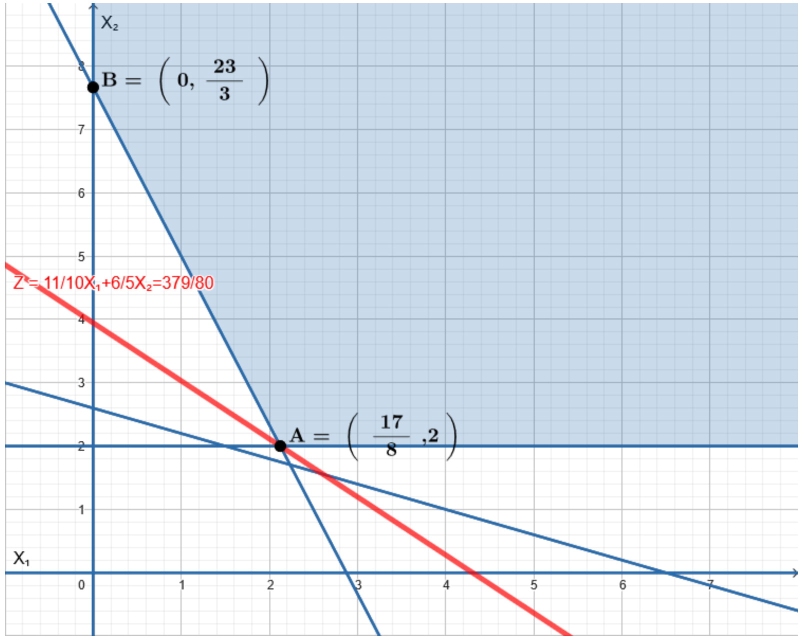
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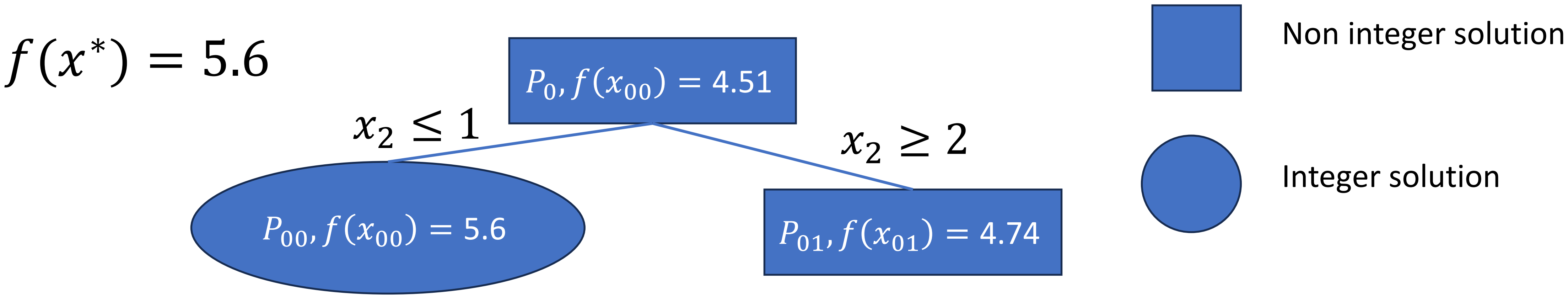
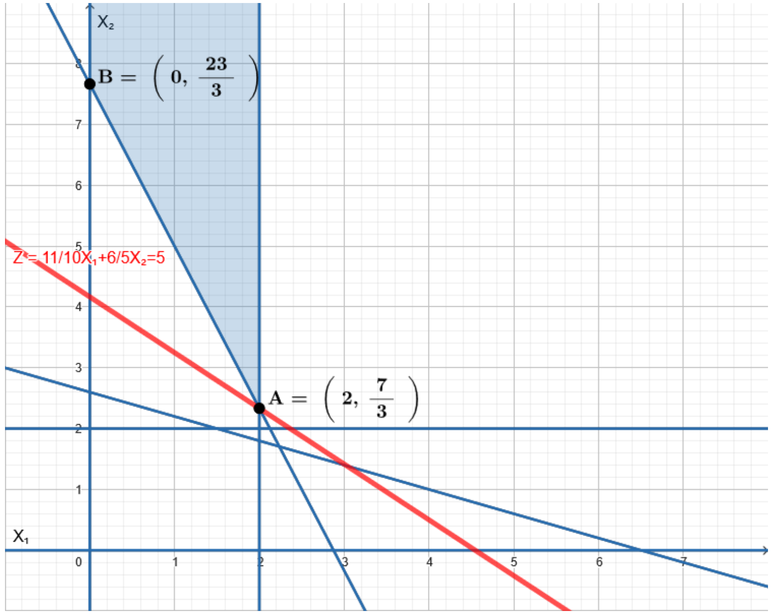
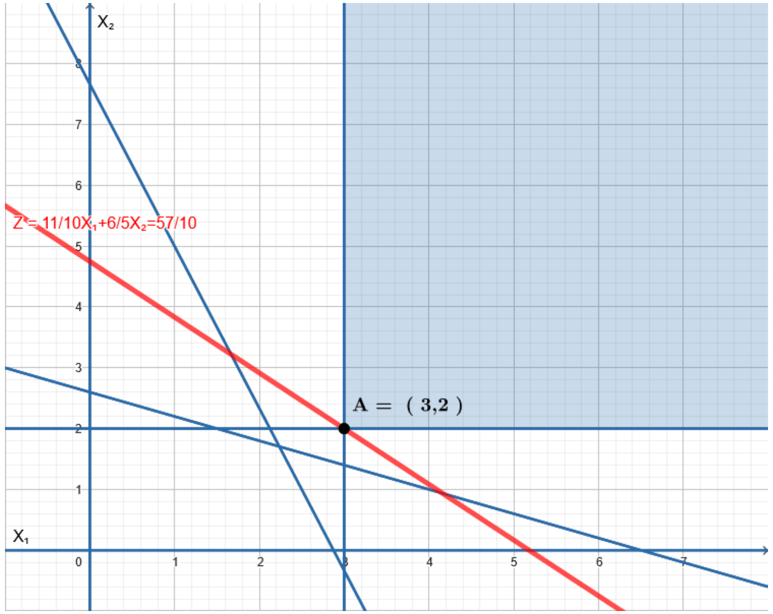

$f(x^*) = 5.6$
$P_0, f(x_{00}) = 4.51$
$x_2 \leq 1$
$x_2 \geq 2$
$P_{00}, f(x_{00}) = 5.6$
$P_{01}, f(x_{01}) = 4.74$
Non integer solution
Integer solution

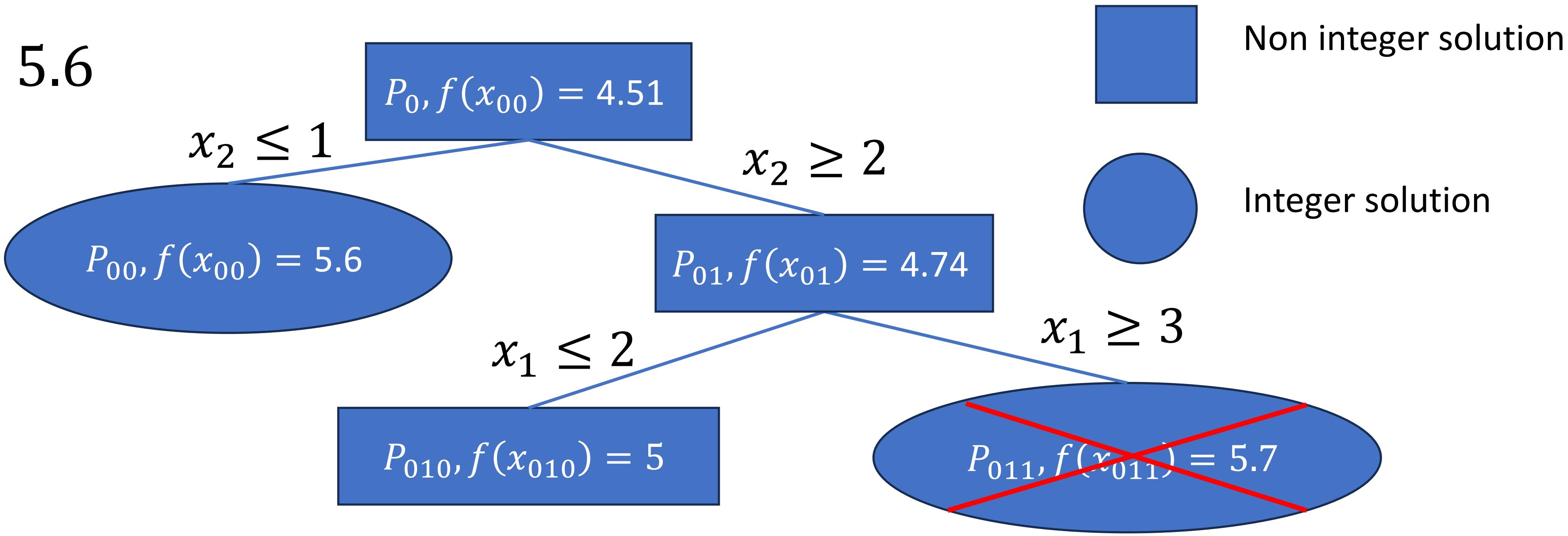
Divide step 2

- The best solution of P_{01} is X1=2.13 , X2=2.
- Divide P_{01} according to variable x_1. Solve graphically the subproblems P_{010} and P_{011}.





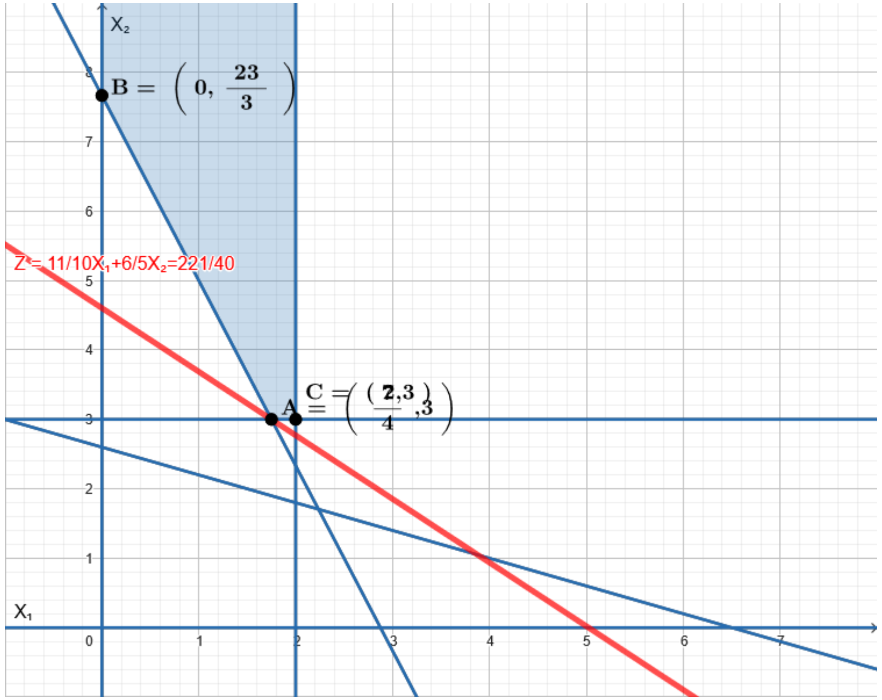
$f(x^*) = 5.6$

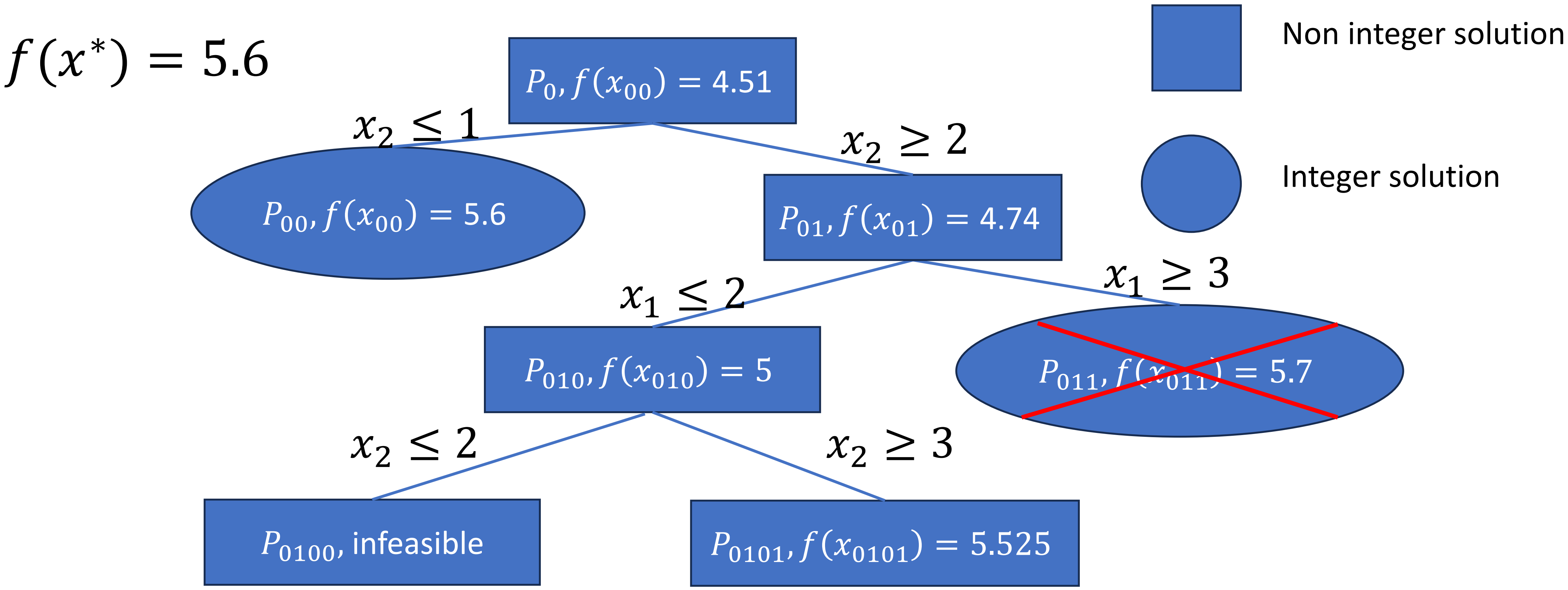


$P_0, f(x_{00}) = 4.51$
$x_2 \leq 1$
$x_2 \geq 2$
$P_{00}, f(x_{00}) = 5.6$
$P_{01}, f(x_{01}) = 4.74$
$x_1 \leq 2$
$x_1 \geq 3$
$P_{010}, f(x_{010}) = 5$
$P_{011}, f(x_{011}) = 5.7$
Non integer solution
Integer solution

Divide step 3

- The best solution of P_{010} is X1=2 , X2=2.33.
- Divide P_{011} according to variable x_2. Solve graphically the subproblems.

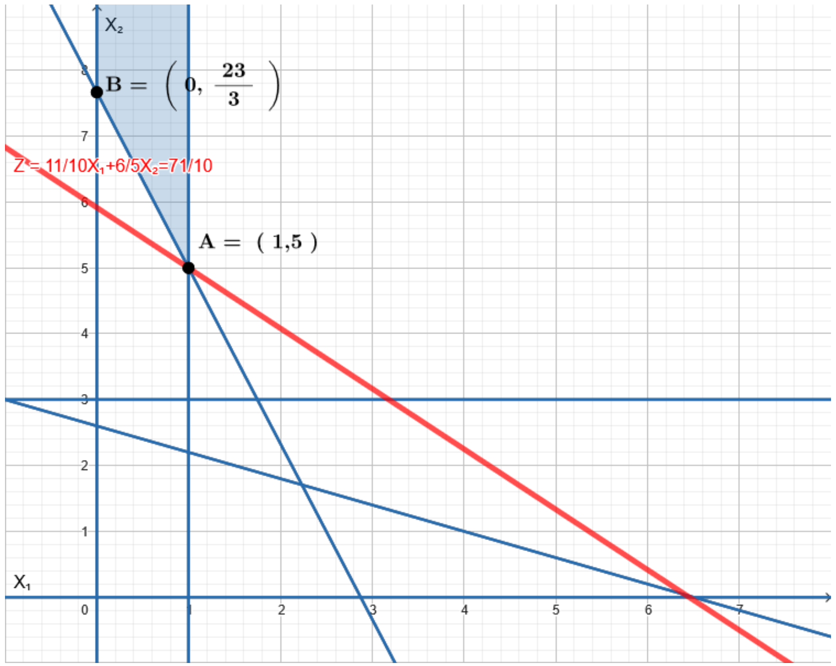


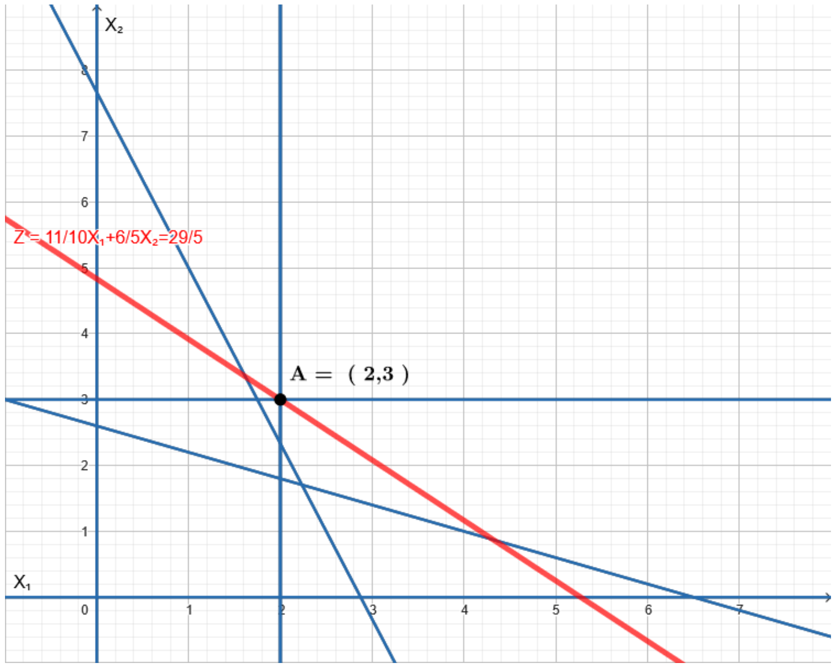


$f(x^*) = 5.6$
$P_0, f(x_{00}) = 4.51$
$x_2 \le 1$
$x_2 \ge 2$
$P_{00}, f(x_{00}) = 5.6$
$P_{01}, f(x_{01}) = 4.74$
$x_1 \le 2$
$x_1 \ge 3$
$P_{010}, f(x_{010}) = 5$
$P_{011}, f(x_{011}) = 5.7$
$x_2 \le 2$
$x_2 \ge 3$
P_{0100}, infeasible
$P_{0101}, f(x_{0101}) = 5.525$
Non integer solution
Integer solution

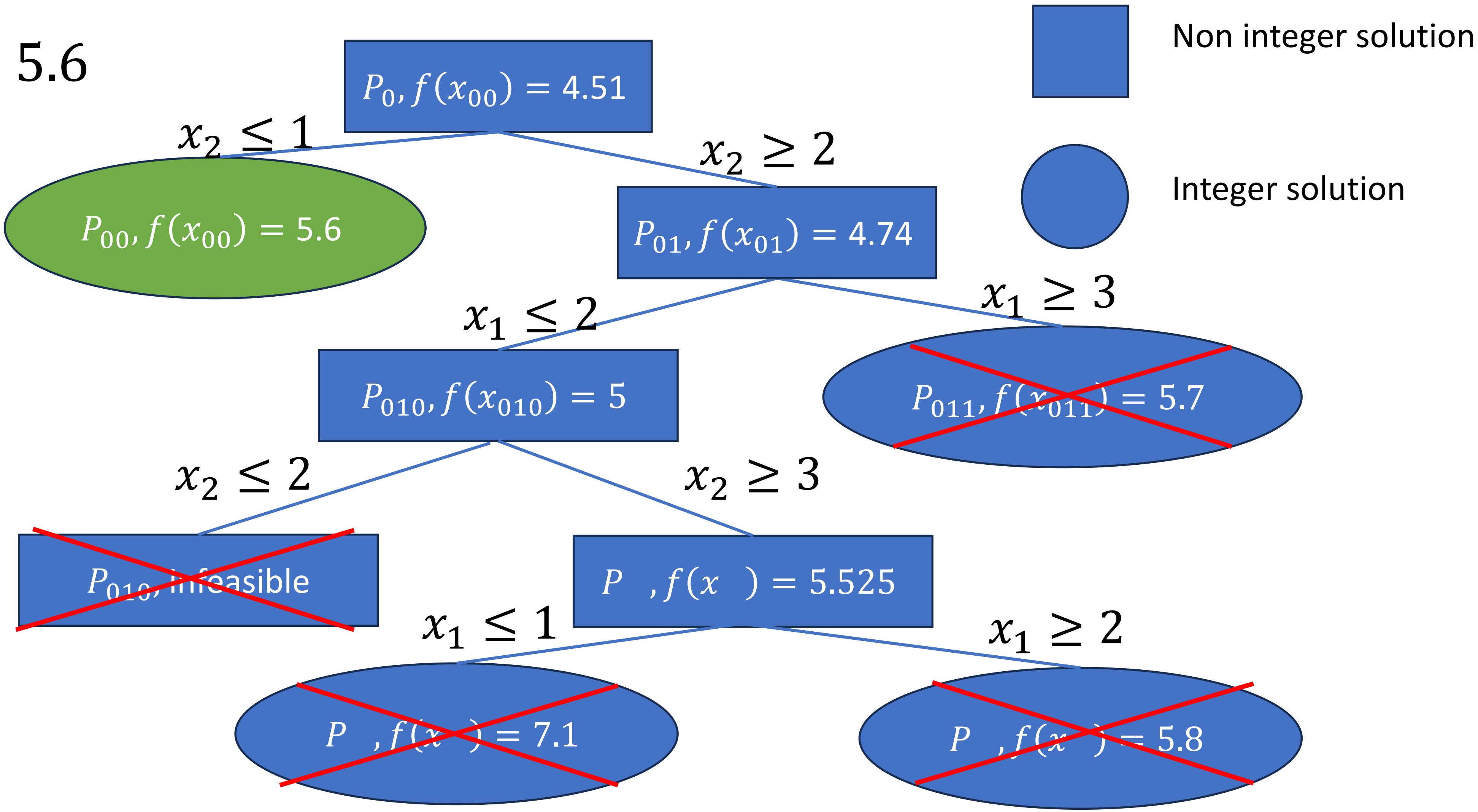
Divide step 4

- The best solution is X1=1.75 , X2=3.
- Divide P according to variable x_1. Solve graphically the subproblems.



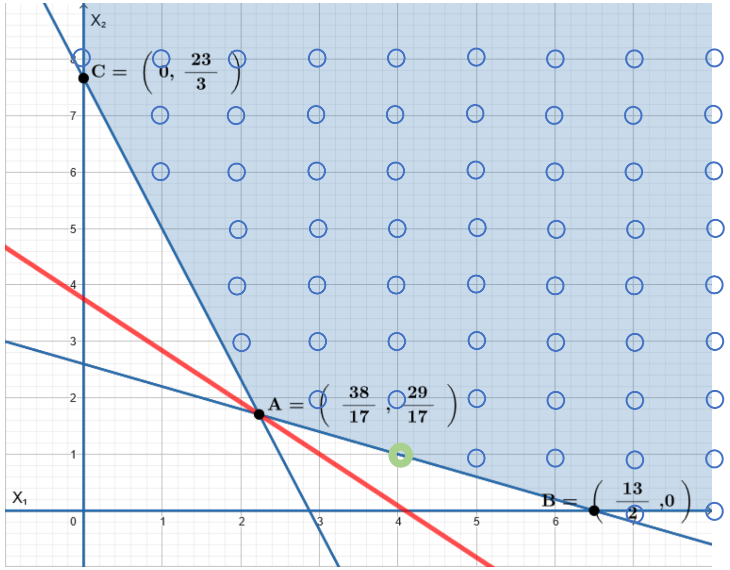


$f(x^*) = 5.6$



$P_0, f(x_{00}) = 4.51$
$x_2 \leq 1$
$x_2 \geq 2$
$P_{00}, f(x_{00}) = 5.6$
$P_{01}, f(x_{01}) = 4.74$
$x_1 \leq 2$
$x_1 \geq 3$
$P_{010}, f(x_{010}) = 5$
$P_{011}, f(x_{011}) = 5.7$
$x_2 \leq 2$
$x_2 \geq 3$
P_{010}, infeasible
$P \ , f(x \ ) = 5.525$
$x_1 \leq 1$
$x_1 \geq 2$
$P \ , f(x \ ) = 7.1$
$P \ , f(x \ ) = 5.8$
Non integer solution
Integer solution

Optimal solution graphically



○ Set of integer solutions

● Optimal solution

Example

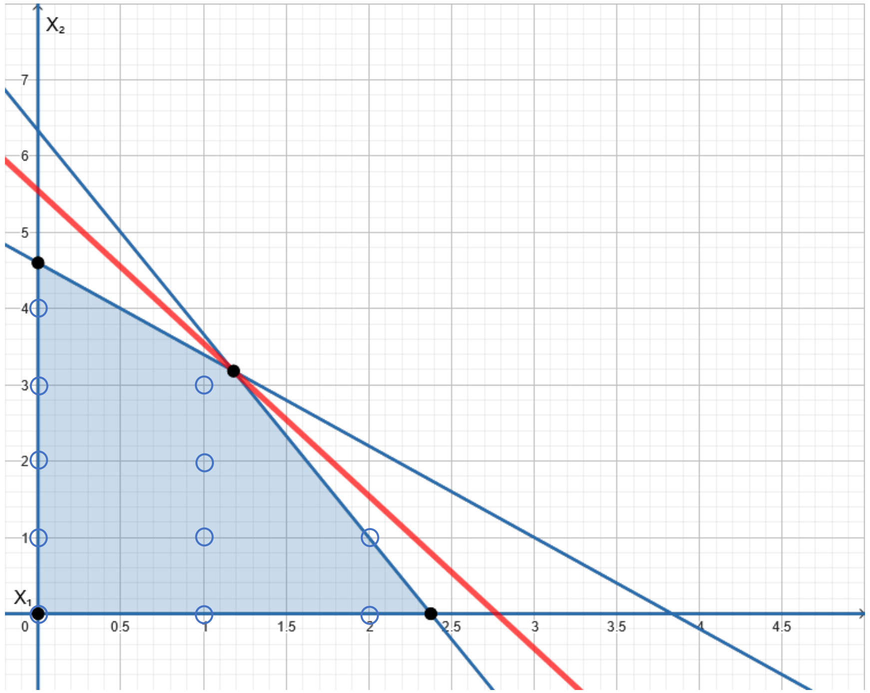
maximize $z = 2x_1 + x_2$

subject to $6x_1 + 5x_2 \leq 23$

$8x_1 + 3x_2 \leq 19$

$x_1, x_2 \geq 0$

$x_1, x_2 \in \mathbb{N}$



○ Set of integer solutions

University of Luxembourg

Merci | Thank you | Danke

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